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On Lamé's Functions with Complex Parameters

BY

J. L. Sharma

The object of the present paper is to consider the solution of Laplace's equation in ellipsoidal coordinates when the parameters assume complex values, and to modify under this condition the expansions of functions in Lamé's polynomials and their products. Cohn * in his doctorate dissertation has discussed under the same conditions the solutions of the equation

$$\frac{\partial^2 f}{\partial \epsilon^2}+\frac{\partial^2 f}{\partial \zeta^2}+n(n+1)(\mu^2-\nu^2)f=0,$$

and mainly the development of functions in Lamé's products by a purely algebraical method based on a limiting process which is hardly applicable in the case of the multiplicity of the roots being higher than two. An attempt is here made, purely on analytical basis, to obtain results in a more general case by a much simpler method.

The consequences arising from the complex values of the parameters are explained in §1. The scope and nature of the problem are stated in the next article. We derive in §3 an important property of the Lamé's functions when B is a multiple root. We study at first the Lamé's functions for values of B different from those satisfying the fundamental equation, and later on we make B approach a multiple root of the equation . These results are utilized in §4 to build up the solutions of the Laplace's equation and to prove their linear independence. In §5, we deduce the solutions of the equation of the surface harmonics which is discussed by Cohn. In §§6-7, we form

* Fritz Cohn, *Über Lamésche Funktionen mit komplexen Parametern*, Inaugural-Dissertation zur Erlangung der Docktorwürde, von der philosophischen Fakultät der Albertus-Universität zu Königsberg 1. Pr. (31, Juli, 1888).

linear combination of the functions obtained in §§3-5 so that they may become orthogonal enabling us to expand functions, the development of which depends on the orthogonality of Lamé's functions or their products. The results of Cohn are derived as an illustration therefrom. In the last article we point out the modification necessary in the Lindemann's expansion of a function of a complex variable z in the Lamé's functions of z.

I express my sincere thanks to Professor Ganesh Prasad for his keen interest in, and kind encouragement of, my work which I undertook at his suggestion.

§1

The Laplace's equation in the ellipsoidal coordinates is

$$\nabla^2 V \equiv \{\wp(u)-\wp(v)\} \frac{\partial^2 V}{\partial w^2} + \{\wp(v)-\wp(w)\} \frac{\partial^2 V}{\partial u^2}$$

$$+\{\wp(w)-\wp(u)\} \frac{\partial^2 V}{\partial v^2} = 0. \qquad (1)$$

This is satisfied by a function of the type

$$V = E(u).\ E(v).\ E(w), \qquad (2)$$

where E satisfies the well known Lamé's equation

$$D^2(E) \equiv \frac{d^2 E}{du^2} - \{n(n+1)\wp(u) + B\}E = 0, \qquad (3)$$

n being a positive integer and B a certain constant. p denoting $\wp(u)$ as usual, this equation has a solution * of one of the forms,

$$K = p^m + a_1 p^{m-1} + \ldots + a_m, \qquad (4)$$

$$M = \sqrt{(p-e_\beta)(p-e_\gamma)}\ (p^{m-1} + b_1 p^{m-2} + \ldots + b_{m-1}), \qquad (5)$$

$$\beta \neq \gamma = 1,\ 2,\ 3$$

* The actual forms of these functions are obtained by me in a paper "On Lamé's Equation" in *Bull. Cal. Math. Soc.*, Vol. XXIII (1931), pp. 101-114.

when $n=2m$, and one of the forms

$$L=\sqrt{p-e_\alpha}\,(p^m+c_1p^{m-1}+\ldots+c_m),\ \alpha=1,\ 2,\ 3 \qquad (6)$$

$$N=p'(p^{m-1}+d_1p^{m-2}+\ldots+d_{m-1}), \qquad (7)$$

when $n=2m+1$; a, b, c and d being functions of B which satisfies the equation $P_n(B)=0$ in cases (4) and (7), and $Q_{e_\alpha}(B)=0$ in the other cases* or in other words which satisfies the joint equation

$$P_n \cdot Q^n_{e_1} \cdot Q^n_{e_2} \cdot Q^n_{e_3}=0. \qquad (8)$$

So long as e_1, e_2, e_3 are real and distinct, it has been shown † that the equation (8) has $2n+1$ distinct roots enabling us to form $(2n+1)$ distinct functions of the aforesaid types They are just necessary to complete the fundamental set ‡ of harmonics of degree n required to furnish the general solution of (1).

When e_1, e_2 and e_3 are not real, it is just possible that some of the roots of the equation 8) may coalesce and thus may cause a deficiency in the number of functions in the fundamental set. It is necessary, therefore, to find a substitute for these missing functions, which should be such as may satisfy the equation (1) and be linearly independent of the other functions of the same degree.

§ 2

We proceed to explain in detail at first how to fill up the gap arising from the coalescence of the roots of $P_n=0$ in the case of the first species and to point out the procedure to be adopted in the case of the other three species. There will be no difficulty in filling up the omitted steps in this account with the aid of the corresponding analysis given in the case of the first species.

The coefficients in the equation for B are functions of two arbitrary constants g_2 and g_3 and a third integral constant n.§ Therefore in general it is possible that for some integral values of n, three roots may coincide; but in special cases it is also possible that more than three roots may coalesce. Hence we proceed on the supposition that r roots become equal. By putting $r=2$ we can derive the results

* $P_n(B)$ and $Q_{e_\alpha}(B)$ are not Legendre's polynomials but certain polynomials whose zeros are B's.

† Halphen, *Traité des Fonctions Elliptiques*, t. 4, p. 471 (Paris, 1888).

‡ Whittaker and Watson, *Modern Analysis*, p. 889.

§ See my paper, *l.c.*

obtained by Cohn. This consideration can be extended without any difficulty to the case of Lamé's functions of higher orders where the coefficients are functions of more than two constants.*

§ 3

3.1. In order to deal with the case of the first species we construct a formal expression of the solution of (3) in a slightly different manner, where the parametric constant B is supposed different from the roots of the equation (8).

Consider the expression

$$y=p^m+a_1p^{m-1}+\ldots+a_{m-1}p+a_m, \tag{9}$$

where a's have the values given by the m equations

$$B+a_1(8m-2)=0,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(m-r+2)(m-r+1)g_3a_{r-2}+\tfrac{1}{2}(m-r+1)(2m-2r+1)g_2a_{r-1}+Ba_r$$
$$+(r+1)\{8m-2(2r+1)\}a_{r+1}=0,$$
$$r=0,\ 1,\ 2,\ldots,m-1,$$

a's with a negative suffix to be taken as zero and $a_0=1$. It should be noted that the forms of a's as functions of B are the same as in the case of Lamé's functions but the value of B in this case is quite arbitrary.†

The effect of applying Lamé's operator, namely,

$$D^2\equiv\left[\frac{d^2}{du^2}-n(n+1)\wp(u)-B\right]$$

to (9) is

$$\left[\frac{d^2}{du^2}-n(n+1)\wp(u)-B\right]y\equiv P_n(B), \tag{10}$$

* Heine, *Theorie der Kugelfunctionen und der Verwandten Functionen*, Theil III, p. 449.

† See my paper, "On Lamé's Equation," *l.c.*

where P_n is the same function of B which in the case of Lamé's functions of the first species gives the values of B.

Regarding y as a function of two independent variables u and B, and differentiating partially with respect to B both the sides of (10), we get,

$$\frac{\partial}{\partial B}\left[\frac{\partial^2}{\partial u^2}-n(n+1)\wp(u)-B\right]y \equiv \frac{dP_n}{dB}. \tag{11}$$

Now y is a polynomial in B, since all the a's are polynomials. It is uniformly continuous in u along with all its successive derivatives in a region excluding the origin and its congruent points, because it is a sum of a finite number of integral powers of $\wp(u)$ which are all uniformly continuous along with their successive derivatives in the same region. Therefore we can change the order of differentiation and get from (11),

$$\left[\frac{\partial^2}{\partial u^2}-n(n+1)\wp(u)-B\right]\frac{\partial y}{\partial B}-y=\frac{dP_n}{dB}. \tag{12}$$

Differentiating r times with respect to B, we get,

$$\left[\frac{\partial^2}{\partial u^2}-n(n+1)\wp(u)-B\right]\frac{\partial^r y}{\partial B^r}-r\frac{\partial^{r-1}y}{\partial B^{r-1}}=\frac{d^rP_n}{dB^r}. \tag{13}$$

Let us suppose now that $P_n=0$ has m_1 roots equal to B_1, m_2 roots equal to B_2, m_3 roots equal to B_3 and so on, such that $m_1+m_2+\ldots=m+1$. Then

$$\left[\frac{d^sP_n}{dB^s}\right]_{B=B_r}=0, \text{ for } s=0, 1, 2, \ldots, m_r-1.$$

Therefore substituting these values of B in (10)—(13) we have the

THEOREM I. *If $P_n=0$ has m_r roots equal to B_r where*

$$\sum_1^t m_r=m+1,$$

then
$$y_{s,r}=\left[\frac{\partial^s y}{\partial B^s}\right]_{B=B_r}, \tag{14}$$

for $s=0, 1, \ldots, m_r-1$, *and* $r=1, 2, \ldots, t$, *satisfy the differential equation*

$$\left[\frac{\partial^2}{\partial u^2}-n(n+1)\wp(u)-B_r\right]y_{s,r}-sy_{s-1,r}=0, \quad (13, a)$$

where y_{0r} *is the Lamé's function of the first species corresponding to* B_r. Thus, in all, we have again $(m+1)$ distinct functions of the first species.

3.2. In the cases of the functions of the other species we take

$$y=(p-e_1)^{k_1}(p-e_2)^{k_2}(p-e_3)^{k_3}(p^m+a_1p^{m-1}+\ldots\ldots a_m), \quad (15)$$

where k_1, k_2, k_3 are either 0 or $\frac{1}{2}$ and $\frac{1}{2}n=m+k_1+k_2+k_3$.

The a's are the same functions of B as the co-efficients of the function of corresponding species are of the roots of $Q_{e_\alpha}=0$ $(\alpha=1, 2, 3)$.

The Lamé's operator D^2 applied out on y will give

$$D^2(y)\equiv(p-e_1)^{k_1}(p-e_2)^{k_2}(p-e_3)^{k_3}Q_{e_\alpha}(B)\equiv\psi(B), \text{say}, \quad (16)$$

in place of (10). The rest of the procedure is similar to the one given above.

When any one of the equations $P_n=0$ or $Q_{e_\alpha}=0$ has multiple roots, the above consideration gives us as many new functions as disappear because of the multiplicity of the roots. Hence in all we shall have $(2n+1)$ functions.

3.3. We proceed now to establish the

THEOREM II. *All these* $2n+1$ *functions are linearly independent.*

In the first place if such a relation existed involving functions of the different species it is obvious that by suitable changes of the signs of the radicals $\sqrt{p-e_\alpha}$ we could obtain other relations which, on being combined by addition or subtraction with the original relation, would give rise to two or more linear relations, each of which involved functions restricted not merely to be of the same species but also of the same type.

Let one of these latter relations, if it exist be

$$a_1 y_{01} + a_2 y_{11} + \ldots + a_{m_1} y_{m_1-1,1} + a_{m_1+1} y_{02} + \ldots + a_q y_q = 0, \qquad (17)$$

$$(a_r \neq 0)$$

among q functions of the same species.

Operate upon the identity $(q-1)$ times with the operator

$$\frac{d^2}{du^2} - n(n+1)\wp(u).$$

The results of the successive operations are

$a_1 B_1^s y_{01} + a_2 (B_1^s y_{11} + s B_1^{s-1} y_{01}) + \ldots + a_q B_q^s y_q = 0$; $s = 1, \ldots, q-1$.

Eliminating a's, we get after simplification

$$\begin{vmatrix} 1 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \\ B_1 & 1 & 0 & \ldots & B_2 & 1 & 0 & \ldots & B_q \\ B_1^2 & 2B_1 & 2 & \ldots & B_2^2 & 2B_2 & 2 & \ldots & B_q^2 \\ B_1^3 & 3B_1^2 & 6B_1 & \ldots & B_2^3 & 3B_2^2 & 6B_2 & \ldots & B_q^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ B_1^{q-1} & l_1 B_1^{q-2} & l_2 B_1^{q-3} & \ldots & B_2^{q-1} & l_1 B_2^{q-2} & l_2 B_2^{q-3} & \ldots & B_q^{q-1} \end{vmatrix} = 0$$

where $l_1 = q-1$, $l_2 = (q-1)(q-2)$.

But this is a function of the differences only, *viz.*,

$$(B_q - B_1)^{m_1} (B_q - B_2)^{m_2} (B_1 - B_2)^{m_1 + m_2} \ldots$$

and therefore cannot be zero unless the B's are equal which is contrary to the hypothesis.

Hence the postulated relation (17) cannot exist.

§ 4

We now substitute a function of the form $V = y(u).y(v).y(w)$ in (1), where y is the same as in (9); then we get,

$$\nabla^2 V \equiv \sum_{uvw} y(u) y(v) \{\wp(u) - \wp(v)\} \frac{\partial^2 y(w)}{\partial w^2}$$

$$\equiv P_n(B)[\sum y(u) y(v) \{\wp(u) - \wp(v)\}]$$

$$\equiv P.\ S, \text{ say.} \qquad (18)$$

Differentiating it r times with respect to B, we have,

$$\frac{\partial^r}{\partial B^r}\nabla^2 V \equiv P.\frac{\partial^r S}{\partial B^r}+r\frac{dP}{dB}\frac{\partial^{r-1}S}{\partial B^{r-1}}+\ldots+S\frac{d^rP}{dB^r}.$$

Now V is continuous throughout any finite region of the B-plane, since it is a polynomial in B and it is uniformly continuous along with all its successive derivatives in the respective regions of u, v and w, excluding, the origin and its congruent points. Hence, changing the order of differentiation, we have,

$$\frac{\partial^r}{\partial B^r}\nabla^2 V=\nabla^2\frac{\partial^r V}{\partial B^r}\equiv P\frac{\partial^r S}{\partial B^r}+\ldots+S\frac{d^rP}{dB^r}. \tag{19}$$

If B_1 is a root of $P_n=0$, (18) shows that

$$V=[y(u)y(v)y(w)]_{B=B_1}$$

will be a solution of (1); and if it is repeated r times, then (19) shows that

$$\left[\frac{\partial^s V}{\partial B^s}\right]_{B=B_1} \quad (s=1, 2,\ldots,r-1)$$

will also satisfy (1). Hence we get the

THEOREM III. *If the roots of* $P_n=0$ *are as stated in Theorem I, then*

$$V_{s,r}\equiv\left[\frac{\partial^s V}{\partial B^s}\right]_{B=B_r} \qquad \begin{cases} s=1, 2,\ldots,m_r-1.\\ r=1, 2,\ldots, t.\end{cases}$$

satisfy the Laplace's equation.

Thus, for m_1 coalescing roots, we have m_1 functions given above. Exactly in the same way we can fill up the gap caused by the coincidence of roots in the case of harmonics of any other species. Hence, in all we shall have again $(2n+1)$ functions required to complete the fundamental set.

These functions are linearly independent, for, if not, by giving particular values to v and w, we will have a linear relation in functions of Theorem I, which is impossible by Theorem II.

We stop here to illustrate the above discussion by an example. Let $n=4$, the equation for B is then

$$B^3-52g_2B+560g_3=0.$$

Its two roots will be equal if

$$-2\left(\frac{52g_2}{3}\right)^{\frac{3}{2}}+560g_3=0,$$

and they will be equal to $\sqrt{\frac{52g_2}{3}}$.

The functions corresponding to this value of B are

$$y = p^2 - \sqrt{\frac{52g_2}{3}} \cdot \frac{p}{14} - \frac{74}{840}\, g_2 \text{ and } -\frac{p}{14} + \frac{\sqrt{52g_2}}{140\sqrt{3}}$$

and the corresponding solutions are :

$$V_1 = \prod_{u,v,w} \left[\wp(u)^2 - \sqrt{\frac{52g_2}{3}} \cdot \frac{\wp(u)}{14} - \frac{37}{420}\, g_2 \right]$$

$$\text{and } V_2 = \sum_{u,v,w} \left[\left(-\frac{p}{14} + \sqrt{\frac{52g_2}{3}} \cdot \frac{1}{140} \right) \left(q^2 - \sqrt{\frac{52g_2}{3}}\, \frac{q}{14} - \frac{37g_2}{420} \right) \right.$$

$$\left. \times \left(r^2 - \sqrt{\frac{52g_2}{3}}\, \frac{r}{14} - \frac{37}{420}\, g_2 \right) \right],$$

where $q = \wp(v)$ and $r = \wp(w)$.

It is easy to see by substitution that

$$\nabla^2 V_2 = 0.$$

§ 5

We come now to the problem of Cohn. The equation of surface ellipsoidal harmonics is

$$\frac{\partial^2 f}{\partial \epsilon^2} + \frac{\partial^2 f}{\partial \zeta^2} + n(n+1)(\mu^2 - \nu^2) f = 0, \tag{20}$$

or, in Weierstrassian elliptic co-ordinates,

$$\vartheta^2 f \equiv \frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial u^2} + n(n+1)\{\wp(u) - \wp(v)\} f = 0. \tag{20.1}$$

Putting $f = y(u)y(v)$, where y denotes the same function as (9), we get,

$$\vartheta^2 f \equiv y(u)\, \frac{\partial^2 y(v)}{\partial v^2} - y(v)\, \frac{\partial^2 y(u)}{\partial u^2} + n(n+1)\{\wp(u) - \wp(v)\} y(u) y(v)$$

$$\equiv P_n(B)\{y(u) - y(v)\} \equiv P.H, \text{ say.} \tag{21}$$

Just as above, differentiating with respect to B, we get,

$$\vartheta^2\, \frac{\partial^r f}{\partial B^r} \equiv P\, \frac{\partial^r H}{\partial B^r} + \dots\dots + H\, \frac{\partial^r P}{\partial B^r}. \tag{22}$$

Now, putting B equal to a multiple root of $P=0$, we deduce the THEOREM IV. *If the roots of* $P_n=0$ *are as stated in Theorem I, then*

$$f_{sr}^{n}=\left[\frac{\partial^s f}{\partial B^s}\right]_{B=B_r} \qquad \begin{cases} s=0,1,\ldots,m_r-1 \\ r=1,2,\ldots,t \end{cases} \tag{23}$$

satisfy the equation (20), *and it is obvious that they are* $(m+1)$ *in number.*

These solutions can be derived from §4 by putting w=a constant, the resulting functions being sums of several of the above functions.

Example: $V_2=\left[\dfrac{d\{y(u)y(v)y(w)\}}{dB}\right]_{B=B_1}$

Regarding w as constant and differentiating, we get,

$$V_2=f_2=\left[w_1\ \frac{d[y(u)y(v)]}{dB}+w_2[y(u)y(v)]\right]_{B=B_1}$$

$$=w_1 f_{11}+w_2 f_{01}.$$

which is a sum of two functions of (23),

In this way again we obtain $(2n+1)$ functions to form a complete fundamental set, which are linearly independent as is obvious from §3.

§ 6

It has been shown* that

$$f(p)=\sum_0^{\sigma} g_s K_s^n(p),$$

where the degree of f is less than or equal to σ.

Even if the roots of $P_n=0$ coincide, we have as in §3, $(\sigma+1)$ linearly independent functions of the class K which are sufficient to enable us to find the values of g's by comparing the co-efficients of different powers of p. The same holds for functions of the other classes.

Further, it has been shown † that

$$f(p)=\sum_{s=0}^{2n} g_s E_s^n(p)$$

* *Cf.* Heine, *l.c.*, (60) and Lindemann, *Math. Annalen*, Bd. 19, p. 327 (5).

† *Cf.* Heine, *l.c.*, p. 371.

$$\text{where}\quad g_s = \frac{\displaystyle\int_{\omega_1}^{\omega_2} \mathrm{E}_s^n(p) f(p)\, du}{\displaystyle\int_{\omega_1}^{\omega_2} \{\mathrm{E}_s^n(p)\}^2 du}.$$

But if B_1 is a multiple root, it can be seen from (10) and (12) that*

$$\int_{\omega_1}^{\omega_2} [\mathrm{E}_1^n(p)]^2 du = 0;$$

this will make g_1 infinite, while $f(p)$ is a finite function.

* Here it should be noticed that g_2, g_3 are not necessarily real, and hence in general, ω_1 and ω_2 are complex. To illustrate it we give an example when $n=4$. In the above E_1 stands for $[y]_{\mathrm{B}=\mathrm{B}_1}$. Let

$$\mathrm{I} = \int_{\omega_1}^{\omega_2} y^2 du = \int_{\omega_1}^{\omega_2} \left(p^2 - \sqrt{\frac{52 g_2}{3}} \cdot \frac{p}{14} - \frac{74}{840} g_2 \right)^2 du.$$

It can be easily shown that

$$\int p^4 du = \frac{1}{840} \left\{ \frac{25}{2} g_2^2 (\omega_2 - \omega_1) + 120 g_3 (\eta_1 - \eta_2) \right\},$$

$$\int p^3 du = \frac{1}{20} \left\{ 3 g_2 (\eta_1 - \eta_2) + 2 g_3 (\omega_2 - \omega_1) \right\},$$

$$\int p^2 du = \frac{g_2}{12} (\omega_2 - \omega_1),$$

$$\int p\, du = (\eta_1 - \eta_2).$$

Hence

$$\mathrm{I} = (\omega_2 - \omega_1) \left[\frac{25}{2.840} g_2^2 + \frac{2 g_3}{20} \left(-2 \sqrt{\frac{52}{3} g_2} \cdot \frac{1}{14} \right) + \frac{g_2}{12} \left\{ \frac{1}{14^2} \frac{52 g_2}{3} - \frac{2.37}{420} g_2 \right\} \right.$$

$$\left. + \left(\frac{74 g_2}{840} \right)^2 \right] + (\eta_1 - \eta_2) \left\{ \frac{g_3}{7} - \frac{3 g_2}{140} \cdot \sqrt{\frac{52}{3} g_2} + \frac{1}{7} \sqrt{\frac{52}{3}} \cdot \frac{37}{420} g_2^{\frac{3}{2}} \right\}$$

$$= 0,$$

since, by the condition of the equality of roots, the coefficients of $(\eta_1 - \eta_2)$ and $(\omega_2 - \omega_1)$ separately vanish.

Henceforward we shall call the functions corresponding to a root as *members of a group.* There shall correspond only one member to a simple root. At first we establish the

THEOREM V. *If ϕ denotes a product of two functions each chosen from a different group, then*

$$\int_{\omega_1}^{\omega_2} \phi du = 0$$

Proof :

From (10) we have,

$$\left\{ \frac{\partial^2}{\partial u^2} - n(n+1)\wp(u) - \mathrm{B} \right\} y_{\mathrm{B}} = \mathrm{P}(\mathrm{B}),$$

also
$$\left\{ \frac{\partial^2}{\partial u^2} - n(n+1)\wp(u) - \beta \right\} y_{\beta} = \mathrm{P}(\beta).$$

Hence

$$(\mathrm{B}-\beta)\int_{\omega_1}^{\omega_2} y_{\mathrm{B}}\, y_{\beta} du - \mathrm{P}(\mathrm{B})\int_{\omega_1}^{\omega_2} y_{\beta} du + \mathrm{P}(\beta)\int_{\omega_1}^{\omega_2} y_{\mathrm{B}}\, du = 0.$$

Differentiating it with respect to B, r times, we have,

$$(\mathrm{B}-\beta)\int_{\omega_1}^{\omega_2} \frac{\partial^r (y_{\mathrm{B}})}{\partial \mathrm{B}^r} y_{\beta} du + r\int_{\omega_1}^{\omega_2} \frac{\partial^{r-1}(y_{\mathrm{B}})}{\partial \mathrm{B}^{r-1}} y_{\beta} du$$

$$- \frac{d^r \mathrm{P}(\mathrm{B})}{d\mathrm{B}^r}\int_{\omega_1}^{\omega_2} y_{\beta} du + \mathrm{P}(\beta)\int_{\omega_1}^{\omega_2} \frac{\partial^r (y_{\mathrm{B}})}{\partial \mathrm{B}^r} du = 0.$$

Differentiating it p times again with respect to β, we have,

$$(\mathrm{B}-\beta)\int_{\omega_1}^{\omega_2} \frac{\partial^r (y_{\mathrm{B}})}{\partial \mathrm{B}^r} \frac{\partial^p (y_{\beta})}{\partial \beta^p} du - p\int_{\omega_1}^{\omega_2} \frac{\partial^r (y_{\mathrm{B}})}{\partial \mathrm{B}^r} \frac{\partial^{p-1}(y_{\beta})}{\partial \beta^{p-1}}$$

$$+ r\int_{\omega_1}^{\omega_2} \frac{\partial^{r-1}(y_{\mathrm{B}})}{\partial \mathrm{B}^{r-1}} \frac{\partial^p y_{\beta}}{\partial \beta^p} du - \dots = 0.$$

Putting $B=B_1$ and $\beta=B_2$ in the above identities and simplifying the results, with the help of identities for lower values of p and r, we get,

$$\int_{\omega_1}^{\omega_2}\left[\frac{\partial^r y_B}{\partial B^r}\right]_{B=B_1}\left[\frac{\partial^p y_\beta}{\partial \beta^p}\right]_{\beta=B_2} du=0,\quad \begin{array}{l} r\leq m_1-1\\ p\leq m_2-1\end{array} \tag{24}$$

$$i.e., \int_{\omega_1}^{\omega_2}\phi\, du=0.$$

From equation (13) we derive

$$\int_{\omega_1}^{\omega_2}\left[\frac{\partial^r y}{\partial B^r}\ \frac{\partial^p y}{\partial B^p}\right]_{B=B_t} du=0,\quad r\neq p\leq m_t-2 \tag{25}$$

and that

$$\int_{\omega_1}^{\omega_2}\left[\frac{\partial^r y}{\partial B^r}\right]^2_{B=B_t} du=0,\qquad r\leq m_t-3, \tag{26}$$

except when $m_t=2$, in which case $\int_{\omega_1}^{\omega_2}\left[y^2\right]_{B=B_1} du=0.$

It shows that the members of the same group are not orthogonal. But a little consideration will show that they can be altered so as to retain that property.

Let

$$y_{r,m_r}=a\left[\frac{\partial^{m_r-1} y}{\partial B^{m_r-1}}\right]_{B=B_r}$$

such, that $\displaystyle\int_{\omega_1}^{\omega_2}(y_{r,m_r})^2 du=1.$

Now, let

$$y_{r,m_r-t}=\sum_{s=1}^{s=t+1} a_s\left[\frac{\partial^{m_r-s} y}{\partial B^{m_r-s}}\right]_{B=B_r}$$

and determine a's such that

$$\int_{\omega_1}^{\omega_2} y_{r,m_r-t}\ y_{r,m_r-p}\, du \begin{array}{l}=0 \quad \text{for } p=0,1,2\ldots t-1\\ =1 \quad \text{for } p=t.\end{array}$$

The a's are completely determined since (25) and (26) do not hold for all values of r and p. Thus, we will have a new group y_r, the members of which are orthogonal and normal among themselves and also with members of the other groups constructed in the same way. Thus we have a new set of $2n+1$ functions which is orthogonal and normal and enables us to determine the g's completely in

$$f(p)=\sum_{s=0}^{2n} g_s y_s$$

by the formula

$$g_s=\frac{\int_{\omega_1}^{\omega_2} f(p)y_s\,du}{\int_{\omega_1}^{\omega_2} (y_s)^2\,du}.$$

§ 7

If we use the equation (20) in place of (20.1) we will obtain $(2n+1)$ linearly independent functions of μ, ν in place of the $(2n+1)$ Lamé's products $E(\mu)E(\nu)$ corresponding to the real values of the parameters. They are just sufficient in number to enable us to find a relation between them and the spherical surface harmonics. Heine's expansions *

$$C_\pi=\sum_{s=1}^{\sigma+1} g^\pi_s K_s(\mu)K_s(\nu),$$

$$C_i=\sum_{s=1}^{n-\sigma} g^i_s L_s(\mu)L_s(\nu),$$

and other two expansions for S_π and S_i, which hold for real values of the parameters b and c, will still remain valid provided we use, in the places of the missing functions, their substitutes found in §5.

We proceed now to expand an arbitrary function of two variables

* *Cf.* Heine, *l.c.*, p. 376.

From the equations (22) and Theorem IV, we see that

$$(m-n)(m+n+1)\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}[\wp(u)-\wp(v)]\, f^n_{sr} f^m_{pt}\, du\, dv$$

$$=\int_{\omega_2}^{\omega_3}\left[f^n_{sr}\frac{\partial f^m_{pt}}{\partial u}-f^m_{pt}\frac{\partial f^n_{sr}}{\partial u}\right]_{\omega_1}^{\omega_2} dv$$

$$-\int_{\omega_1}^{\omega_2}\left[f^n_{sr}\frac{\partial f^m_{pt}}{\partial v}-f^m_{pt}\frac{\partial f^n_{sr}}{\partial v}\right]_{\omega_2}^{\omega_3} du$$

$$=0,$$

since each integrand becomes zero.

Therefore

$$\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\, \underset{sr}{\overset{n}{f}}\ \underset{pt}{\overset{m}{f}}\, du dv=0, \text{ if } m\neq n. \tag{27}$$

For $m=n$,

$$\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\, \underset{pt}{\overset{n}{f}}\ \underset{rs}{\overset{n}{f}}\, du\, dv$$

$$=\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\wp(u)\left[\frac{\partial^p f^n}{\partial B^p}\right]_{B=B_t}\left[\frac{\partial^r f^n}{\partial B^r}\right]_{B=B_s} du\, dv$$

$$-\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\wp(v)\left[\frac{\partial^p f^n}{\partial B^p}\right]_{B=B_t}\left[\frac{\partial^r f^n}{\partial B^r}\right]_{B=B_s} du\, dv.$$

Integrating the first expression on the right with respect to v and the second with respect to u and using the results (24–26), we get,

$$\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\, \underset{pt}{f^n}\ \underset{rs}{f^n}\, du\, dv=0,$$

$$\left.\begin{aligned} &\text{for } t\neq s, \quad p\leq m_t-1\\ &\qquad\qquad\ \ r\leq m_s-1\\ &\text{for } t=s, \quad p\leq m_t-2\\ &\qquad\qquad\ \ r\leq m_s-2\end{aligned}\right\} \tag{28}$$

and also

$$\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\left(f^n_{r\,s}\right)^2 du\, dv=0$$

$$\text{for } r\leq m_s-3 \tag{29}$$

except when $m_s=2$, in which case

$$\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\left(f^n_{0\,s}\right)^2 du\, dv=0. \tag{30}$$

We see from (29) that the functions belonging to the same value B_s are not orthogonal but they can be altered, just as in §6, so that their linear combinations are orthogonal and normal. Therefore, if

$$V^n_{m_1,1} = a\; f^n_{m_1,1}$$

and

$$V^n_{m_1-r,1} = \sum_{t=0}^{t=r} a_{t+1}\, f^n_{m_1-t,1},$$

we find that the V's corresponding to the same value of B are orthogonal and normal among themselves and also with members of the other groups corresponding to different values of B and n.

Hence the constants g's * in

$$f(u,v)=\sum_{n=0}^{\infty}\sum_{s=0}^{2n} g^n_s V^n_s$$

are determined completely by

$$g^n_s = \frac{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2} f(u,v)\{\wp(u)-\wp(v)\}V^n_s\,du\,dv}{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}(V^n_s)^2\,du\,dv}.$$

Example: Suppose that there are only two roots equal to B_1 and the rest are distinct and simple. The functions corresponding to B_1 are

$$V^n_2=\left[\frac{\partial f}{\partial B}\right]_{B=B_1} \text{ and } V^n_1=[f]_{B=B_1}+\kappa\left[\frac{\partial f}{\partial B}\right]_{B=B_1},$$

* *Cf.* Heine, *l.c.*, p. 379 (61).

where κ is such that

$$\left[\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2} \frac{\partial f}{\partial B}\cdot\left\{f+\kappa\frac{\partial f}{\partial B}\right\}\{\wp(u)-\wp(v)\}du\,dv\right]_{B=B_1}=0,$$

Therefore $$\kappa=-\frac{\left[\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}.f\frac{\partial f}{\partial B}\,du\,dv\right]_{B=B_1}}{\left[\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}\left[\frac{\partial f}{\partial B}\right]^2du\,dv\right]_{B=B_1}}$$

$$=-\frac{\alpha}{\beta}\quad\text{(using Cohn's notations).*}$$

Now if $F(u,v)=\sum_{n=0}^{\infty}\sum_{s=1}^{2n+1} g_s^n V_s^n,$

$$g_1^n=\frac{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}F(u,v).V_1^n\,du\,dv}{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}[V_1^n]^2\,du\,dv}$$

$$=\frac{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}F(u,v)\left[f+\kappa\frac{\partial f}{\partial B}\right]_{B=B_1}du\,dv}{\int_{\omega_2}^{\omega_3}\int_{\omega_1}^{\omega_2}\{\wp(u)-\wp(v)\}.\left\{2\kappa f\frac{\partial f}{\partial B}+\kappa^2\left(\frac{\partial f}{\partial B}\right)^2\right\}_{B=B_1}du\,dv}\quad\text{by (30)}$$

$$=\frac{A+\kappa B}{-\frac{\alpha^2}{\beta}+\frac{\alpha^2}{\beta^2}\cdot\frac{\beta}{2}}=-2\frac{\beta A-\alpha B}{\alpha^2}.$$

Similarly

$$g_2^n=\frac{2B}{\beta}.$$

Therefore $$F(u,v)=\frac{2}{\alpha^2}(\alpha B-\beta A)\left\{[f]_{B=B_1}-\frac{\alpha}{\beta}\left[\frac{\partial f}{\partial B}\right]_{B=B_1}\right\}$$

$$+\frac{2B}{\beta}\left[\frac{\partial f}{\partial B}\right]_{B=B_1}+\sum_{n=0}^{\infty}\sum_{(s)} g_s^n V_s^n$$

$$=2\frac{\alpha B-\beta A}{\alpha^2}\cdot f_1^n+\frac{2A}{\alpha}\frac{\partial f_1^n}{\partial B_1}+\sum_n\sum_{(s)} g_s^n f_s^n,$$

* Cohn, *l.c.*, p. 30.

since for simple roots $V_s^n = f_s^n = y_s^n(u) y_s^n(v)$.

These results are the same as those obtained by Cohn *

Similarly, we can find out all the co-efficients in

$$P_n(\cos\gamma) = \sum_{s=0}^{2n} g_s f_s(\mu, v) f_s(\mu_1, v_1),$$

§ 8

Proceeding in the same way as Lindemann † has done in the case of real parameters, it is easy to establish in the light of the above modifications that

$$\frac{1}{R} = \sum_{n=0}^{\infty} \left[\sum_{(B_r)} \sum_{p=0}^{m_r - 1} \left(\frac{\partial^p W^n}{\partial B^p} \right)_{B=B_r} \right],$$

where $W^n = E^n(\mu) E^n(\nu) E^n(\rho) E^n(\mu_1) E^n(\nu_1) F^n(\rho_1)$.

By putting $\nu = \nu_1 = b$ and $\mu = \mu_1 = c$, we can derive,

$$\frac{1}{z - z_1} = \sum_{n=0}^{\infty} \left[\sum_{(B_r)} \sum_{p=0}^{m_r - 1} a_n^p \left(\frac{\partial^p X_n}{\partial B^p} \right)_{B=B_r} \right]$$

$$\text{or} \quad = \sum_{n=0}^{\infty} \left[\sum_{(B_r)} \sum_{p=0}^{m_r - 1} C_p^n \left(\frac{\partial^p K^n(z)}{\partial B^p} \right)_{B=B_r} \right],$$

where $X_n = [K^n(b) K^n(c)]^2 K(z) F(z_1)$

and C_p^n is a function of b, c, B and z_1.

Finally,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z_1)}{z - z_1} \, dz_1$$

$$= \sum_{0}^{\infty} \left[\sum_{(B_r)} \sum_{p=0}^{m_r - 1} d_n^p \left(\frac{\partial^p K^n(z)}{\partial B^p} \right)_{B=B_r} \right]$$

where $$d_n^p = \frac{1}{2\pi i} \int_C C_p^n f(z_1) \, dz_1$$

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* Cohn, *l.c.*, p. 31.

† *l.c.*, pp. 351-360.

6

Growth and Development of Permutations and Combinations in India

BY

Gurugovinda Chakravarti

The subject of permutations and combinations is a noteworthy contribution of the Hindus to the growth and development of mathematics. Its interest originated in connection with the variation of the Vedic metres in a very early age, not in the 12th century with Bhāskara II as has been supposed by Smith.* In the *Chandaḥsūtra* ("the rules of metres") of Piṅgala (before 200 B.C.), a work on Vedic metres, have been found specific rules for the calculation of the possible number of even, semi-even and uneven metres, in a group with a given number of syllables in a quadrant which are nothing other than the calculations of the permutations and combinations.†

After it was once conceived, its principles seem to have had a wide application in the different spheres of Hindu life. Thus for instance, in the medical work of Suśruta, written some six centuries before the Christian era we find that the combinations of the six tastes taken one at a time, two at a time, etc., and all at a time, have been correctly given as 63.‡ In the literature of the Jainas the *Bhagavatīsūtra* (300 B.C.) abounds in instances of speculation about the different philosophical sub-categories that can arise out of the combination of *n* fundamental categories taken one at a time, two at a time, three at a time or more at a time.§ There are similar calculations of the groups that can be formed out of the different instruments of senses (*karaṇas*),|| or of the selections that can be made out of a number of males, females and eunuchs,¶ or of the permutations and combinations in various other

* D. E. Smith, *Hist. of Math.*, (1921), Vol. II, p. 526.

† See post.

‡ *Suśruta Saṁhitā*, Ch. LXIII, *Rasabheda-vikalpādhyāya*.

§ Sūtra, 314.

|| *Ibid*, viii. 5.

¶ *Ibid*, viii. 8. (*s*. 314).

things.* In all the cases the results are given as could be arrived at with the help of the formulae

$$^{n}C_{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.\ 2.\ 3\ldots r},$$

$$^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1).$$

The principles were applied in perfumery too. Varāhamihira (505 A.D.) in the chapter on perfumes of his *Bṛhat Saṁhitā*, has said that "an immense number of perfumes can be made from sixteen substances taken in one, two, three or four proportions"† and has correctly given the number of perfumes resulting from 16 ingredients (being mixed in all proportions) as 174720.‡ The method of calculation he has followed is as follows :—

"Each drug taken in one proportion, being combined with three "others in two, three or four proportions, successively makes "six sorts of scents. Likewise when taken in two, three or "four proportions as in this manner four substances combined "in different proportions yield 24 perfumes, so too, the other "tetrads. Hence the sum will be 96. If a quantity of 16 "substances is varied in four different ways, the result will "be 1820. Since this quantity combined in four ways admit of "96 variations, the number 1820 must be multiplied by 96. The "product will be total of possible combinations of perfumes."§

Varāhamihira made wide applications of this principle in his astrological treatise known as *Bṛhajjātaka* to calculate the number of conjunctions of planets.¶ He has laid down the rule : || "There are 31 varieties of *Anuphā* and 31 of *Sunaphā Yoga* and 180 of *Durudhurā*." That the number of conjunctions given in the rule are quite correct so far as number of possible combinations is concerned, is evident from the definitions of these three conjunctions. The conjunction named *Anuphā*, he says, takes place when one of the five planets (Mars, Mercury, Jupiter, Venus or Saturn) occupy the twelfth house from

* *Ibid*, ix. 32. (s.871-4). See also *Jambudvīpa-prajñapti*, xx. 4, 5, *Anuyogadvāra-Sūtra*, sūtras 76, 96, 126.

† *Bṛhat Saṁhitā*, Ch. 77, Rule 13 and 14. For the translation see Kern's works, Vol. II, p. 87.

‡ *Ibid*, Rule 17.

§ *Ibid*, Rules 18, 19, 20 and 21.

¶ Ch. XIII.

|| *Bṛhajjātaka*, Ch. XIII, Rule 4.

the moon. The *Sunaphā* conjunction is the same as the *Anuphā* except that the planets occupy the second house from the moon. The *Durudhurā* again takes place when these planets occupy the second as well as the twelfth houses from the moon.*

Nomenclature and Terminology.

The Hindu name for the subject is *Bhaṅga* and *Vikalpa Gaṇita.* The word *Vikalpa* can be traced from the period of Suśruta (600 B.C.). But its earliest use, as one of the several topics for discussions in mathematics, is traceable only from the time of the Jaina canon *Sthānāṅga-sūtra* (*c.* 300 B.C.).†

The corresponding Hindu expression used for the modern "taken one at a time," "taken two at a time" etc., as found in the *Bhagavatī-sūtra* ‡ are *Eka saṁyoga* (taken one at a time); *Dvika saṁyoga* (taken two at a time); *Trika saṁyoga* (taken three at a time).

Rules of Combination.

At the begining, the possible varieties of combination among given things was found out by actually representing the combinations graphically. The earliest evidence of an actual calculation of the combination of different things is met with in the medical treatise of Suśruta, where the number 15 by combining the six tastes, taken two at a time, have been found in this way: §

1. Combinations of Sweet.	(*i*)	Sweet and Acid.	
	(*ii*)	,, ,, Saline.	
	(*iii*)	,, ,, Pungent.	
	(*iv*)	,, ,, Bitter.	
	(*v*)	,, ,, Astringent. =	5
2. Combinations of Acid.	(*i*)	Acid and Pungent.	
	(*ii*)	,, ,, Bitter.	
	(*iii*)	,, ,, Astringent.	
	(*iv*)	,, ,, Saline. =	4

* *Ibid*, Rule 3.

† Sūtra 747.

परिकम्म ववहारो रज्जु रासी कला सवमे य।
आवं तावति वग्गो घनो त तह वग्ग वग्गो विकप्पो त ॥

‡ *Bhagavatī-sūtra*, sūtra 314.

§ *Suśruta Saṁhitā*, Ch. LXIII, *Rasabheda-vikalpādhyāya.*

3.	Combinations of Saline.	(*i*)	Saline and Pungent.		
		(*ii*)	,, ,, Bitter.		
		(*iii*)	,, ,, Astringent.	=	3
4.	Combination of Pungent.	(*i*)	Pungent and Bitter.		
		(*ii*)	,, ,, Astringent.	=	2
5.	Combination of Bitter and Astringent.			=	1

The total number, thereby, comes to $5+4+3+2+1=15$.

This method was followed in the Jaina canons and even in later times when easier and shorter mathematical rules were known. Varāhamihira (505 A.D.) used it as an alternative method. This was known as the method of *Loṣṭra Prastāra*. He says, "Fill up the (first place) by any one of the given number until it (the number) is exhausted. Then fill up the second place and others in the same way."*

With the mathematical rules however we become acquainted from the 3rd century B.C. only. Piṅgala laid down a rule to find the number of combinations of *n* syllables, taken one at a time, two at a time, etc.,...and all at a time. His aphorism, "Then the sum total"† being very short can be understood only with the commentary of Halāyudha, who has explained the rule as follows:

"Draw a square. Beginning at half of the square, draw two other "similar squares below it, below the two, three other squares "and so on. By putting one in the first square, the marking

* इच्छाविकल्पैः क्रमशोऽभिनीय
नीते निवृत्तिः पुनरन्यनीतिः ॥
Bṛhat Saṁhitā, loc. cit. Ch. 77, Rule 22.

† परे पूर्णमिति। viii, 34. Halāyudha explains as follows :—

अनेन एकद्व्यादिलघुक्रियासिद्ध्यर्थं यावदभिमतं प्रथमप्रस्तारवत् मेरुप्रस्तारं दर्शयति। उपरिष्टादेकं चतुरस्रकोष्ठं लिखित्वा तस्याधस्तात् उभयतोऽर्धनिष्क्रान्तं कोष्ठकद्वयं लिखेत्, तस्याप्यधस्तात्त्रयं तस्याप्यधस्ताच्चतुष्टयमेवं यावदभिमतं स्थानमिति मेरुप्रस्तारः। तस्य प्रथमे कोष्ठे एकसंख्याम् व्यवस्थाप्य लक्षणमिदं प्रवर्तयेत्। तत्र द्विकोष्ठायां पङ्क्तावुभयोः कोष्ठयोरेकैकमङ्कं दद्यात्। ततस्तृतीयायां पङ्क्तौ पर्यन्तकोष्ठयोरेकैकमङ्कम् दद्यात्। मध्यमकोष्ठे तूपरिकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं निवेशयेदिति पूर्णशब्दार्थः। चतुर्थ्यां पङ्क्तावपि पर्यन्तकोष्ठयोरेकैकमङ्कम् स्थापयेत्। मध्यमकोष्ठयोस्तूपरिकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं त्रिसंख्यारूपं स्थापयेत्। उत्तरत्राप्ययमेव न्यासः। तत्र द्विकोष्ठायां पङ्क्तावेकाक्षरस्य प्रस्तारः। ... ... तृतीयायां पङ्क्तौ द्व्यक्षरस्य प्रस्तारः। चतुर्थ्याम् पङ्क्तौ त्र्यक्षरस्य प्रस्तारः।

"should be started. In the two squares of the second line put "1 in each. In the third line put 1 in the two squares at the "ends and in the middle square the sum of the digits in the "two squares lying above it. In the fourth line put 1 in the "two squares at the ends. In the middle ones put the sum total "of the digits in the two squares above each. Proceed on in "this way. Of these the second line gives the combinations "with one syllable....This third line gives the combinations "with two syllables and etc."

This method, known as the *Meru Prastāra* * may be represented graphically as follows :—

No. of Syllables.										Total No. of Combinations.
	1									
1	1	1								2
2	1	2	1							4
3	1	3	3	1						8
4	1	4	6	4	1					16
5	1	5	10	10	5	1				32
6	1	6	15	20	15	6	1			64
7	1	7	21	35	35	21	7	1		128
8	1	8	28	56	70	56	28	8	1	256

Etc. Etc. Etc.

It is quite evident that in the above in each line, *i.e.*, with any number of syllables, the total number of combinations have been found out by first of all finding out the number of combinations

* *Piṅgala-Chandaḥ-sūtra*, Ed. Bhagabaticharan Smrititirtha, p. 162.

possible by taking one at a time, two at a time, etc., and taking all at a time and then by taking their sum or to put it mathematically, the total number of combinations

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n.$$

Another formula of noteworthy interest that Piṅgala used in laying down the rule is

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}.$$

This is quite clear from Piṅgala's aphorism, "Then the sum total" just quoted. The commentator has clearly said in explaining the above aphorism that the true import of the aphorism is that "the sum of the digits in the two squares just above is to be taken. This is why 'sum total' has been used."* To explain this with a concrete example of Piṅgala, let us take the case of 10 in the line corresponding to five syllables (in the *Meru Prastāra*). According to the rule, 10 is to be obtained by adding 4 and 6. Now, 10 represents ${}^5C_2$, 4 represents ${}^4C_1$ and 6 represents ${}^4C_2$. $4+6=10$ is therefore ${}^4C_1 + {}^4C_2 = {}^5C_2$ which is a particular case of the general formula,

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}.$$

In later times Varāhamihira (505 A.D.) borrowed the principle of the Meru Prastāra to lay down his rule for finding the combinations. His rule though slightly different in form, from that of Piṅgala, is however fundamantally the same as that of the latter. He says,† "It is said that the numbers are obtained by adding each with the one which is past the one in front of it, except the one in the last

* . 34. viii परे पूर्णमिति । The commentator says : मध्यमकोष्ठे तूपरिकोष्ठद्वयांकमेकीकृत्य पूर्णं निवेशयेदिति पूर्णशब्दार्थः ।

† *Bṛhat-Saṁhitā*, Ch. 77, stanza 22.

पूर्व्वेण पूर्व्वेण गतेन युक्तं
स्थानं विनान्त्यं प्रवदन्ति संख्याम् ।

place." The commentator, Bhaṭṭotpala, has illustrated the rule with an example.*

To find the combinations of sixteen things taken one at a time, two at a time, three at a time, etc.

	Taken two at a time.	Taken four at a time.
16		
15	120	
14	105	
13	91	1820
12	78	1365
11	66	1001
10	55	715
9	45	495
8	36	330
7	28	210
6	21	126
5	15	70
4	10	35
3	6	15
2	3	5
1	1	1

Bhaṭṭotpala further tells us that the combinations may be obtained by another means: "Putting down (the figures) once in the reverse order, put them below again in the direct order. (In finding the final result) multiply the numbers in the direct process, *i.e.*, from left to right and divide by the corresponding numbers below."† This, however, is in substance the other method already stated and comes obviously to the formula

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\; r}\dots \textit{i.e.,} \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

Exactly the same rule appears in the *Gaṇita-Sāra-Saṁgraha* of Mahāvīra‡ and the *Līlāvatī* of Bhāskara §

* *Bṛhat-Saṁhitā*, Ed. MM. Sudhakara Dvivedi, pp. 952-953.

† *Bṛhajjātaka*, p. 228.

‡ *Gaṇita-sāra-saṁgraha*, Ch. VI, Rule 218.

§ *Līlāvatī*, Ed. H. C. Banerjee, VI, Rules 110-12.

Piṅgala contributed two other formulæ to combinations. One* of them is that in a group of m syllables, the limit of semi-even metres is $(2^m)^2-2^m$ and the other † is that the limit of uneven is $(2^{2m})^2-2^{2m}$.

Permutations.

We have already seen that by the time when the *Bhagavatī-sūtra* was written, some method of finding out the permutation of certain things was known. But the definite formulation of any mathematical rule is traceable only from the time of *Anuyogadvāra-sūtra.* We give below the free translation of a passage from the work relevant to this.‡ "What is direct order? Dharmāstikāya, Adharmāstikāya, Ākāśāstikāya, Jīvāstikāya, Pudgalāstikāya and Samaya—this is the direct order. What is the reverse order? (Read all these six from the reverse order, *i.e.*, from Samaya to Dharmāstikāya). What is Ananupūrvī or mixed order? In this series, in which the first term is one, the common increase is one and the number of terms is six; multiply all the terms one after anothor and deduct two. This gives Ananupūrvī of the mixed order." In other words the total number of permutations of six things is given by $1\times2\times3\times4\times5\times6$. This is nothing but a particular case of the general formula $1.\ 2.\ 3.\ \ldots\ \ldots\ (n-1)\ n$. The direction for deducting 2 is to give the number of permutations less the direct and reverse orders. There are several other instances of this in the *Anuyogadvāra-sūtra.*§

A Sanskrit verse giving exactly the rule that if n be the number of different things given, then the total number of permutations that can be made with them will be given by $1.2.3.\ \ldots\ \ldots\ (n-1)n$, is quoted by Śīlāṅka.|| Bhāskara too repeated the rule.¶ Two other verses,

* V. 3. समं तावत् ज्ञतः ज्ञतमर्धसमं.

† V. 4 and 5. विषमयं, राश्यूनम् ।

‡ Rule 97. सेकिं तम् पुव्वाणुपुव्वी ? धम्मात्थिकाय आगासात्थिकाए जीवात्थिकाए पोग्गलत्थिकाए अद्धासमए सेतम् पुव्वाणुपुव्वी । से किं तं अनाणुपुव्वी ? एयाए चेव एगादिआए एगुत्तरिआए छगच्छगयाए सढीए अन्नमन्नब्भासो दुरूणो, सेतं अनाणुपुव्वी ॥

§ See also Rules 103, 115, 116 and others.

|| *Vide* his commentary on the *Sūtra,-Kṛtāṅga-Sūtra*, *Samayādhyāyan*, *Anuyogadvāra*, verse 28.

एकाद्या गच्छपर्यन्ताः परस्परसमाहताः ।
राशयस्तद्धि विज्ञेयं विकल्पगणिते फलम् ॥

¶ *Līlāvatī*, *loc. cit.*, XIII, 272.

one * in Ardhamāgadhī and the other † in Sanskrit, both having the same purpose of dictating a rule for finding the actual spread or representation (*Prastārānapnopāya*). The rules are, however, cryptic and can be understood only with the help of the commentary of Śīlāṅka who has explained the whole thing with an illustration. This has already been thoroughly discussed by Dr. Datta in his "Jaina School of Mathematics" in the following words : ‡

"The Sanskrit verse may be rendered as follows :—

"The total number of permutations being divided by the last term the quotient should be divided by the next. They should be placed accordingly by the side of the initial term in the calculations of permutations and combinations."

"Let there be r number of things, $a_1, a_2, a_3, \ldots \ldots a_r$. Then the total number of permutations that can be made with them will be by the previous rule, $1.2.3 \ldots \ldots (r-1)r$ or $r\,!$. The number of permutations which can have any particular thing, say, for its initial digit (ādi) will be $r!/r$ that is $(r-1)!$. So put a_r in the begining of $(r-1)!$ number of grooves. Similarly put a_{r-1} in the begining of another $(r-1)!$ grooves and so on. Again amongst the first series of grooves, the number of sub-grooves that can have a_{r-1} after a_r will be $(r-1)!/(r-1)$ or $(r-2)!$. Place a_{r-1} after a_r in those sub-grooves. The number of sub-gooves that can have a_{r-2} after a_r will be $(r-2)!$ and put it after a_r, in those sub-grooves. Similarly with a_{r-3}, $a_r-4, \ldots \ldots a_1$. Again amongst the sub-grooves that can have any other particular thing in the third place will be $(r-2)!/(r-2)$ or $(r-3)!$ and it should be placed in those cases. Proceeding step by step in this way in a systematic manner we can find out all the different permutations of things."

Bhāskara made some other valuable contributions to the subject. He dictated rules to find :

(1) the sum or amount of permutations with specific numbers §

* पुव्वानुपुव्वि हेट्ठा समयाभेएण कुणजहाजेट्ठं ।
उवरिमतुल्लं पुरओ नसेज्ज पुव्वक्कमो सेसे ॥

† गणितेऽन्त्यविभक्ते तु लब्धं शेषैर्विभाजयेत् ।
आदावन्ते च तत् स्थाप्यं विकल्पगणिते क्रमात् ॥

‡ *Bull. Cal. Math. Soc.*, XXI, (1929), pp. 135-36.

§ *Līlāvatī*, *loc. cit.*, XIII, 267.

$$\frac{\underline{|n}}{n} \times \text{sum of digits} \times (10^{n-1} + \ldots + 10 + 1).$$

(2) In the above case when two or more digits are alike : suppose p of them to be alike and q of them also to be alike : then the rule is*

$$\frac{\underline{|n}}{n\,\underline{|p}\,\underline{|q}} \times \text{sum of the digits} \times (10^{n-1} + \ldots + 10 + 1).$$

(3) The permutations with indeterminate digits for a definite sum and a specific number of place :†

$$\frac{(s-1)(s-)\ldots\{s-(n-1)\}}{1.2\ldots(n-1)}.$$

Bull. Cal. Math. Soc., Vol. XXIV, No. 2 (1932)

* *Ibid*, XIII, 270.

† *Ibid*, XIII, 274.

On the Summation of Infinite Series of Legendre's Functions

(Part I)

BY

Durgaprasad Banerjee

After reading articles 11 and 20 of Dr. Ganesh Prasad's *Spherical Harmonics, Part II*, I felt encouraged to add to the list of such infinite series of Legendre's functions $P_n(\cosh\sigma)$ or $Q_n(\cosh\sigma)$ with non-integral n, as admit of being summed up in terms of elliptic or simpler functions.

As is known, the first successful attempt at the summation of such series was made by Dr. Ganesh Prasad * and then his work was continued by Mr. N. G. Shabde.†

Many of the series given below are believed to be new and they are starred. There are also a number of other series, which, although they can be obtained as deductions from the results of Dr. Ganesh Prasad or by methods similar to those used by him, have been added as they are interesting.

My thanks are due to Dr. Ganesh Prasad for the interest he has taken in the paper.

*(1) Let $K_p = K_p(\cosh\psi) = P_{-\frac{1}{2}+p}(\cosh\psi)$;

then

$$\frac{1}{2}K_0 - \frac{1}{2}K_1 + \frac{1}{5}K_2 - \frac{1}{10}K_3 + \frac{1}{17}K_4 - \ldots \ldots \text{ to infinity}$$

$$= \frac{\operatorname{cosech}\pi}{k^{\frac{1}{2}}} E(k'), \text{ for } -\pi < \psi < \pi.$$

* Ganesh Prasad, *Bull. Cal. Math. Soc*, XXIII (1931), pp. 115-124.

† Shabde, *Bull. Cal. Math. Soc.*, XXIII (1931), pp. 155-182.

It is known* that $K_p = \frac{2}{\pi}\int_0^{\psi} \frac{\cos p\phi d\phi}{\sqrt{2(\cosh\psi - \cosh\phi)}}$.

Therefore the

$$\text{series} = \frac{2}{\pi}\int_0^{\psi} \frac{d\phi}{\sqrt{2\cosh\psi - \cosh\phi)}} \left\{ \frac{1}{2} - \frac{1}{2}\cos\phi + \frac{1}{5}\cos 2\phi \right.$$

$$\left. -\frac{1}{10}\cos 3\phi \ldots \ldots \text{ to inf.} \right\} \qquad \ldots \ldots (i)$$

The cosine series † within the crooked brackets equals $\frac{\pi}{2}$ cosech π cosh ϕ, when $-\pi < \phi < \pi$. Therefore (i) equals

$$\frac{2}{\pi} \cdot \frac{\pi}{2\sinh\pi}\int_0^{\psi} \frac{\cosh\phi\, d\phi}{\sqrt{2(\cosh\psi - \cosh\phi)}}$$

$$= \frac{\pi}{2}\,\text{cosech}\,\pi.\quad \overset{\ddagger}{P}_{\frac{1}{2}}(\cosh\psi) = \frac{\text{cosech}\,\pi}{k^{\frac{1}{2}}}\,E(k'),$$

where $\quad 2\cosh\psi = k + \frac{1}{k}$ and $k^2 + k'^2 = 1$.

(2) $\frac{1}{2}(\cosh\pi - 1)K_0 - \frac{1}{2}(\cosh\pi + 1)K_1 + \frac{1}{5}(\cosh\pi - 1)K_2$

$-\frac{1}{10}(\cosh\pi + 1)K_3 + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ to infinity

$= 2\sinh\frac{\psi}{2}$, for $-\pi < \psi < \pi$.

Proceeding as in (1) we find that the series has the sum

$$\frac{2}{\pi}\int_0^{\psi} \frac{d\phi}{\sqrt{2(\cosh\psi - \cosh\phi)}}\left[\frac{1}{2}(\cosh\pi - 1) - \frac{1}{2}(\cosh\pi + 1)\cos\phi\right.$$

$$\left. + \frac{1}{5}(\cosh\pi - 1)\cos 2\phi \ldots \ldots \ldots \text{ to inf.}\right] \qquad \ldots \ldots \ldots (i)$$

* See Hobson, "On a type of spherical harmonics of unrestricted degree, order and argument," *Phil. Trans.* (London), A. Vol. 187 (1896).

† W. E. Byerly, *Fourier's series*, p. 46.

‡ Ganesh Prasad, *Sperical Harmonics*, Part II, p 20.

The cosine series* within the crooked brackets has the sum

$$\frac{\pi}{2} \sinh \phi.$$

Therefore $(i) = \int_0^{\psi} \frac{\sinh \phi d\phi}{\sqrt{2(\cosh \psi - \cosh \phi)}} = 2 \sinh \frac{\psi}{2}$.

*(3) $\frac{1}{2\mu^2} K_0 - \frac{K_1}{\mu^2 - 1^2} + \frac{K_2}{\mu^2 - 2^2} + \frac{K_3}{\mu^2 - 3^2}$ .... to infinity

$= \frac{\pi}{2\mu \sin \mu\pi} K_\mu$, if μ be a real fraction.

The series $= \frac{2}{\pi} \int_0^{\psi} \frac{d\phi}{\sqrt{2(\cosh \psi - \cosh \phi)}} \left[\frac{1}{2\mu^2} - \frac{\cos \phi}{\mu^2 - 1^2} \right.$

$\left. + \frac{\cos 2\phi}{\mu^2 - 2^2} - \quad \ldots \quad \ldots \quad \text{to inf.} \right]$ ... (ii)

The cosine series† $= \frac{\pi}{2\mu} \frac{\cos \mu\phi}{\sin \mu\pi}$.

Therefore $(ii) = \frac{\pi}{2\mu \sin \mu\pi} \cdot \frac{2}{\pi} \int_0^{\psi} \frac{\cos \mu\phi d\phi}{\sqrt{2(\cosh \psi - \cosh \phi)}}$

$$= \frac{\pi K_\mu}{2\mu \sin \mu\pi}.$$

(4). $\frac{1}{2}(e^{\pi} - 1)K_0 - \frac{1}{1+1^2}(e^{\pi}+1)K_1 + \frac{1}{1+2^2}(e^{\pi}-1)K_2$...to infinity

$$= \frac{E(k')}{k^{\frac{1}{2}} \sinh \pi} + 2 \sinh \frac{\psi}{2}.$$

* Byerly, *l.c.*, p. 46.

† Byerly, p. 46.

$$\text{The series} = \frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}\left[\frac{1}{2}(e^{\pi}-1)\right.$$

$$\left.-\frac{1}{1+1^2}(e^{\pi}+1)\cos\phi+\ldots\ldots\text{ to inf.}\right]. \quad \ldots (iii)$$

The cosine series * $= \dfrac{\pi}{2}e^{\phi}$.

$$\text{Therefore } (iii) = \int_0^{\psi}\frac{e^{\phi}\,d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}$$

$$= \frac{E(k')}{k^{\frac{1}{2}}\sinh\pi}+2\sinh\frac{\psi}{2}. \quad \text{[Adding the results of (1) and (2)].}$$

*(5) $\dfrac{K_0}{2}-\dfrac{K_1}{m^2+1^2}+\dfrac{K_2}{m^2+2^2}-\dfrac{K_3}{m^2+3^2}\quad\ldots\quad\ldots$ to infinity

$$= \frac{\pi}{2\sinh m\pi}\,P_{m-\frac{1}{2}}(\cosh\psi), \text{ for } -\pi<\psi<\pi.$$

$$\text{The series} = \frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}\left[\frac{1}{2}-\frac{\cos\phi}{m^2+1^2}\right.$$

$$\left.+\frac{\cos 2\phi}{m^2+2^2}\quad\ldots\ldots\text{ to infinity}\right]. \quad \ldots (iv)$$

The cosine series † $= \dfrac{\pi}{2}\,\dfrac{\cosh m\phi}{\sinh m\pi}$.

$$\text{Therefore } (iv) = \frac{1}{\sinh m\pi}\int_0^{\psi}\frac{\cosh m\phi\,d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}$$

$$= \frac{\pi}{2\sinh m\pi}\,P_{m-\frac{1}{2}}(\cosh\psi). \ddagger$$

* Byerly, p. 46.

† Byerly, *l.c.*, p. 46.

‡ Prasad, *l.c.*, p. 19.

Hence a toroidal function is expanded in a series of conal functions.

(6) $\frac{\cosh 2\pi-1}{2} K_0 - \frac{\cosh 2\pi+1}{2^2+1^2} K_1 + \frac{\cosh 2\pi+1}{2^2+2^2} K_2 - \ldots$ to infinity

$$=\frac{4}{3}(2\cosh\psi+1)\sinh\frac{\psi}{2}.$$

$$\text{The series} = \frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}\left[\frac{\cosh 2\pi-1}{2}\right.$$

$$\left. -\frac{\cosh 2\pi+1}{2^2+1^2}\cos\phi+\ldots\ldots\ldots \text{ to inf.}\right] \quad \ldots \quad (v)$$

The cosine series* $= \frac{\pi}{2}\sinh 2\phi$.

$$\text{Therefore } (v) = \int_0^{\psi}\frac{\sinh 2\phi\, d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}$$

$$=\frac{4}{3}(2\cosh\psi+1)\sinh\frac{\psi}{2}.$$

*(7) $\frac{1}{3}K_1 - \frac{\pi}{16}K_2 + \frac{K_3}{1.3.5} - \frac{K_5}{3.5.7} + \frac{K_7}{5.7.9}\ldots$

to infinity

$$=\frac{1}{8}k^{\frac{1}{2}}F(k'), \text{ for } -\frac{\pi}{2}\leq\psi\leq\frac{\pi}{2}\cdot$$

Now

$\frac{K_3}{1.3.5} - \frac{K_5}{3.5.7} + \frac{K_7}{5.7.9} - \ldots$ to infinity

* Byerly, *l.c.*, p. 46.

$$=\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}\left[\frac{\cos 3\theta}{1.3.5}-\frac{\cos 5\theta}{3.5.7}+\frac{\cos 7\theta}{5.7.9}\right.$$

$$\left.-\ldots\text{to inf.}\right]\ldots\ldots(vi)$$

The cosine series *

$$=\frac{\pi}{8}\cos^2\theta-\frac{1}{3}\cos\theta=\frac{\pi}{16}(1+\cos 2\theta)-\frac{1}{3}\cos\theta.$$

Therefore (vi)

$$=\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}\left[\frac{\pi}{16}(1+\cos 2\theta)-\frac{1}{3}\cos\theta\right]$$

$$=\frac{1}{8}\int_0^{\psi}\frac{d\phi}{\sqrt{2(\cosh\psi-\cosh\phi)}}+\frac{\pi}{16}K_2-\frac{1}{3}K_1$$

$$=\frac{1}{8}\;k^{\frac{1}{2}}F(k')+\frac{\pi}{16}K_2-\frac{1}{3}K_1.$$

Hence $\quad \frac{1}{3}K_1-\frac{\pi}{16}K_2+\frac{K_3}{1.3.5}-\frac{K_5}{3.5.7}+\frac{K_7}{5.7.9}\ldots$

to infinity $=\frac{1}{8}\,k^{\frac{1}{2}}F(k')$.

*(8) Let $Q_{m-\frac{1}{2}}=Q_{m-\frac{1}{2}}(\cosh\psi)$.

$$Q_{-\frac{1}{2}}+\frac{1}{2}Q_{\frac{1}{2}}-\frac{2}{1.3}Q_{\frac{3}{2}}-\frac{2}{2.4}Q_{\frac{5}{2}}-\frac{2}{3.5}Q_{\frac{7}{2}}\ldots$$

to infinity

$$=-2\pi\sinh\frac{\psi}{2}-2\cosh\frac{\psi}{2}\,E\left(k,\frac{\pi}{2}\right),\ \text{where }\cosh\frac{\psi}{2}=\frac{1}{k}.$$

* Macrobert, *Functions of a complex Variable*, p. 285.

Note on the Type of Expanding Universe Recently Proposed by Einstein and de Sitter

BY

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1. The object of the present note is to discuss some properties of the non-static model of the universe, recently proposed by Einstein and de Sitter.* The line-element approximately representing the present state of the universe on the average is taken to be

$$ds^2 = -R^2(dx^2 + dy^2 + dz^2) + c^2dt^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where R is a function of t only and c is the velocity of light.

The expressions for pressure and density as obtained from the field-equations

$$-8\pi T^{\mu}_{\nu} = G^{\mu}_{\nu} - \tfrac{1}{2}g^{\mu}_{\nu}\,(G - 2\lambda)$$

are

$$8\pi p_0 = -\frac{2\ddot{R}}{Rc^2} - \frac{\dot{R}^2}{R^2c^2} + \lambda \quad \ldots \quad \ldots \quad (2)$$

$$8\pi\rho_{00} = \frac{3\dot{R}^2}{R^2c^2} - \lambda. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Einstein and de Sitter have considered the equations with $\lambda = 0$. Here λ is retained for a fuller discussion.

The model may be both expanding or contracting, but it cannot change from the one to the other, as $\dot{R}$ can never vanish. (For, when $\dot{R} = 0$, from (3) if $\lambda \neq 0$, ρ_{00} becomes negative; and if $\lambda = 0$, ρ_{00} vanishes in which case we may also suppose p_0 to vanish, whence from (2) $\ddot{R} = 0$, so that the model becomes perfectly static.)

In the present note, we assume that the model *expands* through a radius $R = R_0$, at the time $t = 0$, the corresponding velocity of expansion being $\dot{R} = \dot{R}_0$.

The first condition that the model should satisfy is the condition of positive pressure and we shall impose later on a further condition that the density does not vanish for any finite value of the radius R.

* Einstein and de Sitter—Proc. Nat. Acad. Sciences (America), Vol. 18, p. 232, 1932.

2. From (2), the first condition $p_0 \geqq 0$ gives

$$\frac{2\ddot{R}}{Rc^2}+\frac{\dot{R}^2}{R^2c^2}-\lambda \leqq 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

Multiplying (4) by the positive quantity $c^2R^2\dot{R}$ (the model is expanding) and integrating between limits $R=R_0$, $\dot{R}=\dot{R}_0$ to $R=R$ $\dot{R}=\dot{R}$, we get

$$R\dot{R}^2-R_0\dot{R}_0^2-\frac{\lambda c^2}{3}(R^3-R_0^3)\leqq 0$$

that is, $$R\left(\dot{R}^2-\frac{\lambda c^2}{3}R^2\right)\leqq R_0\left(\dot{R}_0^2-\frac{\lambda c^2}{3}R_0^2\right).$$

Thus we have

$$\left.\begin{aligned} Rf_R &\leqq R_0 f_{R_0} \\ f_R &\leqq f_{R_0} \end{aligned}\right\} \qquad (A)$$

from which also follows (since $R>R_0$)

where $f_R \equiv \dot{R}^2-\dfrac{\lambda c^2}{3}R^2$.

We conclude that *in the Expanding Universe described by* (1), *the rate of expansion can only be such that the function* Rf_R (*hence also* f_R) *continually decreases.*

Again we have from (A)

$$\dot{R}^2 \leqq \frac{\lambda c^2}{3}R^2+\left(\dot{R}_0^2-\frac{\lambda c^2}{3}R_0^2\right)$$

or, $$dt \geqq \frac{dR}{\sqrt{\left\{\left(\dot{R}_0^2-\frac{\lambda c^2}{2}R_0^2\right)+\frac{\lambda c^2}{3}R^2\right\}}}. \qquad \ldots \qquad (5)$$

Integrating (5) between the corresponding limits $t=0$ to $t=t$ and $R=R_0$ to $R=R$, we get

$$t \geqq \sqrt{\frac{3}{\lambda c^2}}\log\frac{R}{R_0}$$

$$-\sqrt{\frac{3}{\lambda c^2}}\log\frac{\sqrt{\frac{\lambda c^2}{3}}+\frac{\dot{R}_0}{R_0}}{\sqrt{\frac{\lambda c^2}{3}}+\sqrt{\left\{\frac{\lambda c^2}{3}+\frac{1}{R^2}\left(\dot{R}_0^2-\frac{\lambda c^2}{3}R_0^2\right)\right\}}} \qquad (B)$$

$$=\vartheta-T,$$

where ϑ and T denote the first and the second expression respectively in (B).

It can be easily verified that the second term T is positive (argument > 1).

The inequality (B) gives a lower bound of the time taken by the model to expand from a radius R_0 to another R so that the expansion cannot take place in a time less than that given by the right hand side of (B).

Further, if we impose the condition $\rho_{00} \geqq 0$ in (3),

we have $$\dot{R}^2 \geqq \frac{\lambda c^2}{3} R^2 \qquad (6)$$

that is $$\frac{\dot{R}}{R} \geqq \sqrt{\frac{\lambda c^2}{3}}. \qquad \ldots \quad (6')$$

Integrating (6′) between corresponding limits,

$$t \leqq \sqrt{\frac{3}{\lambda c^2}} \log \frac{R}{R_0} \qquad \ldots \quad (C)$$

or, $\leqq \vartheta$.

From (C) it is evident that *every finite radius of the universe will be reached in finite time.*

Relations (B) and (C) show that *the time of expansion, t, from radius R_0 to R lies between the two bounds given by*

$$\vartheta - T \leqq t \leqq \vartheta.$$

A greater lower bound can be found if we start from the inequality,

$$R f_R \leqq R_0 f_{R_0}.$$

This gives $$dt \geqq \frac{dR}{\sqrt{\left\{ \frac{R_0}{R}\left(\dot{R}_0^2 - \frac{\lambda c^2}{3} R_0^2 \right) + \frac{\lambda c^2}{3} R^2 \right\}}} \qquad \ldots \quad (7)$$

So that $$t \geqq \int_{R_0}^{R} \frac{\sqrt{R}\, dR}{\sqrt{\left\{ R_0(\dot{R}_0^2 - \frac{\lambda c_0^2}{3} R_0^2) + \frac{\lambda c^2}{3} R^3 \right\}}}.$$

It is evident on comparing (5) and (7) that this lower bound is greater than the preceding one.

Coming back to (A), it is to be noted that f_R will always remain positive as is evident from (6), and decreasing while R continually increases. But if $\dot{R}$ decreases there will be a finite value of $\dot{R}$ for which f_R vanishes. On the other hand (C) shows that so long as R remains

finite the time of increasing to that radius is also finite, so that f_R can vanish within a finite time and for a finite value of R. This is contrary to the imposed condition ($\rho_{00}>0$). Hence $\dot{R}$ *must continually increase in the expanding universe* (1). So that the expansion will be quicker with time.

3. Let us assume, with Einstein and de Sitter, $\lambda=0$.

Then from (A) we have,

$$R\dot{R}^2 \leq R_0\dot{R}_0^2 \qquad \ldots \quad (D)$$

or in other words, *with* $\lambda=0$, $R\dot{R}^2$ *and consequently also* $\dot{R}$ *continually decreases.*

That $\dot{R}$ is also decreasing continually in this case also follows from Eqn. (2) (with $\lambda=0$). Since in this equation $\ddot{R}$ must be negative in order that the pressure may not be negative.

From (D) it is evident that $\dot{R}$ decreases faster than $\frac{1}{\sqrt{R}}$, but as has been shown before it can never become zero in finite time.

In other words, *the rate of expansion of the universe* (1), (*with* $\lambda=0$) *is continually slowing down but it can never become zero without the universe being completely empty and perfectly static.*

It is to be noted that with $\lambda\neq0$ the expansion grows quicker continually. This is a fundamental difference between the cases $\lambda=0$ and $\lambda\neq0$ for this line element.

Multiplying Eqn. (3) by R^3 and comparing with conditions (A) and (D) we find that both for $\lambda\neq0$ and $\lambda=0$, $\rho_{00}R^3$ continually decreases so that *the density of energy in the universe decreases faster than* $\frac{1}{R^3}$. The vanishing of the density implies the stopping of the expansion.

A lower bound for the time of expansion from R_0 to R, in this case, is given by

$$t \geq \frac{2}{3}\,\frac{R^{\frac{3}{2}}-R_0^{\frac{3}{2}}}{\dot{R}_0\sqrt{R}} \qquad \text{(obtained from D)}$$

but no upper bound is obtainable by the analogous process.

In conclusion, I wish to express my grateful thanks to Prof. N. R. Sen for suggestions and help.

Bull. Cal. Math. Soc., Vol. XXIV, No. 2 (1932).

Principal S. C. Bagchi, M.A., L.L.D.
with compliments.

5

On the summability (C, 1) and strong summability (C, 1) of certain divergent Legendre series

BY

H. P. Banerjea

It is well known * that the Fourier series corresponding to a continuous function is convergent (C, 1). A *direct verification* of this theorem for the case of a continuous function whose Fourier series is divergent at a point was first given † by Fejér. The corresponding theorem for the Legendre series was first proved by Haar ‡ and subsequently discussed by Chapman, § Gronwall, || Lukàcs ¶ and Fejér.** The *direct verification* of the convergence (C, 1) of certain continuous functions whose Legendre series diverges has been taken up in this paper. I prove in Art. 1, that the Legendre series corresponding to the first function given by Lukács †† is convergent (C, 1). In Art. 2, I prove that the second function ‡‡ of Lukàcs also possesses this property. In Art. 3, I show that the series corresponding to both the functions of

* "Untersuchungen über Fouriersche Reihen"—L. Fejér (*Math. Ann.*, Bd. 58, 1904). See also *Comptes Rendus* t. 131 (1900) and t. 134 (1904).

† "Sur les singularités de la série de Fourier des fonctions continues"—L. Fejér (*Annali di L'Ecole normale superieure*, tome 88, 1911).

‡ "Über die Legendresche Reihe"—A. Haar (*Rend. del Circ. Mat. di Palermo*, tomo 32, 1911). See also *Math. Ann.*, Vol. 69, 1910.

§ *Quarterly Journal of Pure and Applied Mathematics*, Vol. 43, 1912; *Math. Ann.* Bd. 72.

|| *Mathematische Annalen*, Bd. 74, 1913.

¶ *Comptes Rendus*, t. 157, 1913; *Math. Zeitschrift* Bd. 14.

** *Mathematische Zeitschrift*, Bd. 24, 1926.

†† *Mathematische Zeitschrift*, Bd. 14, 1922.

‡‡ *Ibid.*

Lukàcs are strongly summable (C, 1). * In Art. 4, I discuss for increasing n the infinitary behaviour of

$$\sum_{m=1}^{n} |s_m - s|^q, \quad \text{for } q > 1,$$

where s_m is the mth partial sum of the Legendre series and s the value of the function at the point considered.

My best thanks are due to Professor Ganesh Prasad for encouragement and interest.

1. *Lukàcs's first function.*

Let
$$\Phi(\theta) = \sum_{n=1}^{\infty} \frac{f_{\nu_n}(\theta)}{n^2}, \qquad \ldots \quad (1)$$

where $f_\nu(\theta) = \sqrt{\nu+1}\ \{P_\nu(\cos\theta) - P_{\nu+2}(\cos\theta)\}\ (1-\cos\theta)$,

and $\nu_n = n^6$.

It has been proved † by Fejér that

$$|\sqrt{\nu+1}\ \{P_\nu(\cos\theta) - P_{\nu+2}(\cos\theta)\}| < C, \quad \ldots \quad (2)$$

for all integral values of ν, C being a constant independent of ν and θ.

Hence

$$\Phi(\theta) \equiv \sum_{n=1}^{\infty} \frac{\sqrt{n^6+1}}{n^2} \{P_{n^6}(\cos\theta) - P_{n^6+2}(\cos\theta)\}(1-\cos\theta),$$

is, on account of the inequality (2), easily seen to be a continuous function of θ, being the sum of an absolutely, and therefore uniformly, convergent series.

* In *Comptes Rendus*, t. 156, 1913, Hardy and Littlewood have proved that under certain restrictions if the Fourier series corresponding to $f(x)$ is convergent (C, 1), it is also strongly summable (C, 1). In *Fundamenta Mathematicae*, Vol. 10, 1927, A. Zygmund proves the corresponding theorem for normalized orthogonal functions. Thus Art. 3 is a direct verification of the theorem, corresponding to Hardy and Littlewood's theorem, for the Legendre series.

† *Mathematische Zeitschrift*, Bd. 24, 1926.

The kth partial sum s_k of the Legendre series of $\Phi(\theta)$ is

$$s_k(\theta) = \sum_{r=0}^{k} \frac{2r+1}{2} P_r(\cos\theta) \int_0^{\pi} P_r(\cos\phi)\, \Phi(\phi) \sin\phi \, d\phi$$

$$= \frac{k+1}{2} \int_{-1}^{1} \Phi(\phi) dx \left\{ \frac{P_k(x) P_{k+1}(\cos\theta) - P_{k+1}(x) P_k(\cos\theta)}{\cos\theta - x} \right\}$$

(putting $x = \cos\phi$)

$$= \frac{k+1}{2} \sum_{n=1}^{\infty} \frac{\sqrt{n^6+1}}{n^2} \int_{-1}^{1} \frac{P_k(x) P_{k+1}(\cos\theta) - P_{k+1}(x) P_k(\cos\theta)}{\cos\theta - x}$$

$$\times \{P_{n^6}(x) - P_{n^6+2}(x)\}(1-x)dx$$

Hence putting $\theta = 0$,

$$s_k(0) = \frac{k+1}{2} \sum_{n=1}^{\infty} \frac{\sqrt{n^6+1}}{n^2} \int_{-1}^{1} \{P_k(x) - P_{k+1}(x)\}\{P_{n^6}(x) - P_{n^6+2}(x)\} dx$$

So we have

$$s_{n^6-1} = -n^6 \cdot \frac{\sqrt{n^6+1}}{n^2} \cdot \frac{1}{2n^6+1},$$

$$s_{n^6} = (n^6+1) \frac{\sqrt{n^6+1}}{n^2} \cdot \frac{1}{2n^6+1},$$

$$s_{n^6+1} = (n^6+2) \frac{\sqrt{n^6+1}}{n^2} \cdot \frac{1}{2n^6+5},$$

$$s_{n^6+2} = -(n^6+3) \frac{\sqrt{n^6+1}}{n^2} \cdot \frac{1}{2n^6+5},$$

for all integral values of n. For all other positive values of k, such that k is not of the forms $n^6 \pm 1$, n^6 and n^6+2, $(n = 1, 2, 3, \ldots)$ the values of $s_k(0)$ are all zero.

Hence, although the values of $s_k(0)$ are either in absolute value very large or zero as k becomes bigger and bigger,

$$\left| \frac{\sum_{n=0}^{k} s_n}{k+1} \right| \leq \sum_{n=1}^{N} \left\{ \frac{-n^6}{n^3} \cdot \frac{\sqrt{n^6+1}}{2n^6+1} + \frac{(n^6+1)}{n^3} \frac{\sqrt{n^6+1}}{2n^6+1} \right.$$

$$\left. + \frac{n^6+2}{n^3} \cdot \frac{\sqrt{n^6+1}}{2n^6+5} - \frac{n^6+3}{n^3} \cdot \frac{\sqrt{n^6+1}}{2n^6+5} \right\} \Big/ (k+1),$$

N being so chosen that $N^6 \leq k+1 < (N+1)^6$.

Thus

$$\left| \frac{\sum_{n=0}^{k} s_n}{k+1} \right| \leq \frac{\sum_{n=1}^{N} \frac{\sqrt{n^6+1}}{n^3} \left\{ \frac{1}{2n^6+1} - \frac{1}{2n^6+5} \right\}}{k+1}$$

$$\leq \frac{\sum_{n=1}^{N} \sqrt{1+\frac{1}{n^6}} \; \frac{1}{n^{11}\left(1+\frac{1}{2n^6}\right)\left(1+\frac{5}{2n^6}\right)}}{k+1}.$$

Here the numerator is for large values of N of the same order as $\Sigma \frac{1}{n^{11}}$, and as this series is absolutely convergent, the numerator tends to a finite limit, say C', where C' is an absolute constant, so that

$$\lim_{k \to \infty} \left| \frac{\sum_{n=0}^{k} s_n(0)}{k+1} \right| = \lim_{k \to \infty} \frac{C'}{k+1} = 0.$$

Thus the divergent Legendre series corresponding to the continuous function $\Phi(\theta)$ given by (1) is convergent (C,1) at $\theta=0$.

2. *Lukàcs's second example.*

Let $$\Phi(\theta) = \sum_{n=2}^{\infty} \frac{\sqrt{n!}}{n^2} \{P_{n!}(\cos\theta) - P_{n!+2}(\cos\theta)\} \qquad \ldots \quad (8)$$

Here we notice that on account of the inequality (2) of the previous article, the function $\Phi(\theta)$ is a continuous function of θ and its expansion in a Legendre series gives

$$\Phi(\theta)=0+0+\frac{\sqrt{2!}}{2^2}P_2(\cos\theta)+0-\frac{\sqrt{2!}}{2^2}P_4(\cos\theta)$$

$$+0+\frac{\sqrt{3!}}{3^2}P_6(\cos\theta)+0-\frac{\sqrt{3!}}{3^2}P_8(\cos\theta)+\dots$$

The partial sum s_k of the first $(k+1)$ terms of this series for $\theta=0$ is either 0 or very large as k becomes bigger and bigger ; nevertheless

$$\frac{\sum_{n=0}^{k} s_n(0)}{k+1} \leq \frac{2\sum_{n=2}^{N}\frac{\sqrt{n!}}{n^2}}{k+1},$$

where N is so chosen that $N! \leq k < (N+1)!$

Thus

$$\frac{\sum_{n=0}^{k} s_n(0)}{k+1} \leq \frac{2\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{k+1}$$

$$\leq \frac{2}{\frac{k+1}{\sqrt{N!}}}\;\frac{\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{\sqrt{N!}}.$$

But we know that if $M_1, M_2, \dots M_n$ be a monotone increasing sequence of positive numbers and if $\sum a_n$ is convergent,

$$\lim_{n\to\infty}\frac{M_1a_1+M_2a_2+\dots+M_na_n}{M_n}=0.$$

Therefore

$$\lim_{k\to\infty}\frac{\sum_{n=0}^{k} s_n(0)}{k+1}=\lim_{k\to\infty}\frac{2}{\frac{k+1}{\sqrt{N!}}}\;\frac{\left\{\frac{\sqrt{2!}}{2^2}+\frac{\sqrt{3!}}{3^2}+\dots+\frac{\sqrt{N!}}{N^2}\right\}}{\sqrt{N!}}$$

$=0$, since $k+1 \geq N!+1$.

Thus in this case, as well, the divergent Legendre series corresponding to the continuous function $\Phi(\theta)$ given by (3) is convergent (C, 1) at $\theta=0$.

3. To consider the strong summability (C, 1) of the Legendre series corresponding to Lukàcs's first function, we have, since $\Phi(0)=0$,

$$\left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| \leq \sum_{n=1}^{N} \left\{ \frac{n^6\sqrt{n^6+1}}{n^3(2n^6+1)} + \frac{(n^6+1)\sqrt{n^6+1}}{n^3(2n^6+1)} \right.$$

$$\left. + \frac{(n^6+2)\sqrt{n^6+1}}{n^3(2n^6+5)} + \frac{\sqrt{n^6+1}\,(n^6+3)}{n^3(2n^6+5)} \right\} \Big/ k+1,$$

where as before $N^6 \leq k+1 < (N+1)^6$.

Therefore

$$\left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| \leq \frac{2\sum_{n=1}^{N} \frac{\sqrt{n^6+1}}{n^2}}{k+1}$$

$$\leq \frac{2\sum_{n=1}^{N} n\sqrt{1+\frac{1}{n^6}}}{k+1}.$$

The numerator here is for large values of N of the same order as $2\sum_{n=1}^{N} n$ *i.e.*, as $N(N+1)$.

Hence

$$\lim_{k\to\infty} \left| \frac{\sum_{n=0}^{k} |s_n-\Phi(0)|}{k+1} \right| = \lim_{k\to\infty} \frac{N(N+1)}{k+1}$$

$$= \lim_{N\to\infty} \frac{N(N+1)}{(N+1)^6} = 0.$$

The series is therefore strongly summable (C, 1).

To discuss the strong summability (C, 1) of the series corresponding to the second function of Lukàcs, we notice that $\Phi(0)$ is zero and that each partial sum is positive or zero. Then since the corresponding series is convergent (C, 1), it is also strongly summable (C, 1).

The same remarks also apply to the case of Gronwall's example (*Math. Ann.*, Vol. 74).

4. To determine the greatest integral value of q such that

$$\frac{\sum_{\nu=0}^{k} | s_k - \Phi(0) |^q}{k+1} \to 0$$

for the first function of Lukàcs, we have

$$\frac{\sum_{\nu=1}^{N-1} K_\nu}{k+1} < \left| \frac{\sum_{\nu=0}^{k} | s_\nu |^q}{k+1} \right| \leq \frac{\sum_{\nu=1}^{N} K_\nu}{k+1},$$

For the second function of Lukàcs, we have

$$\left| \frac{\sum\limits_{n=0} | s_n - \Phi(0) |^q}{k+1} \right| \propto \frac{\sum\limits_{n=2}^{N} \frac{(n\,!)^{\frac{q}{2}}}{n^{2q}}}{k+1},$$

$$\text{where} \quad N! \leq k < (N+1)!$$

$$\propto \frac{\left\{ \frac{(2\,!)^{\frac{q}{2}}}{2^{2q}} + \frac{(3\,!)^{\frac{q}{2}}}{3^{2q}} + \dots \frac{(N\,!)^{\frac{q}{2}}}{N^{2q}} \right\}}{N\,!+1}$$

In the limit the right hand side becomes zero, when $\frac{q}{2}=1$, *i.e.* $q=2$. For $q > 2$, the right hand side is infinite * in the limit. Thus the required greatest value of q is 2.

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9

Concerning two Square Root Methods

BY

RAYMOND GARVER

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*(Communicated by Prof. G. Prasad *)*

The purpose of the present note is to make certain comments concerning the two square root methods discussed in the interesting paper by Dr. Datta, which appeared in the *Bulletin of the Calcutta Mathematical Society*, Vol. 23, 1931, 187-94. Both the methods, of course, have a long history, as Dr. Datta points out, but there are certain facts about them which may be worth mentioning.

One method takes y/x as an approximate value of $\sqrt{N}$, where (x, y) is a solution of the Diophantine equation $Nx^2+1=y^2$. Dr. Datta remarks, on page 188, that the error in this approximation is not greater than $1/2xy$ or $1/2x^2\sqrt{N}$. There is, curiously, a fortunate mistake in this statement; while the error is less than $1/2x^2\sqrt{N}$, it is greater than $1/2xy$. This can be verified at once by writing $\sqrt{N} = \frac{y}{x}\sqrt{1-\frac{1}{y^2}}$ and expanding the right-hand side by the binomial theorem. We thus have an upper and lower limit to the error, and the two are very close to each other. Thus the approximation 49/20

* Dr. Datta writes to Prof. Prasad : " The article of Mr. Raymond Garver should be printed in the *Bulletin*. I am sorry that a mistake crept into my paper. Though my object was purely historical, to show that the Hindu Nārāyana (1860) anticipated the Swiss-German Euler (1782), and not to test the comparative value of that method and Heron's method, yet that simple misstatement should not have been there."

or $2 \cdot 45$ to $\sqrt{6}$ is in error by more than $1/1960$ or $\cdot 0005102$ and is less than $9/800 \cdot 22$ or $\cdot 0005113$. (In obtaining this upper limit $\sqrt{N}$ in the upper limit formula was replaced by $22/9$, which is a little too small. The value $22/9$ was obtained by continued fractions.) We know then that $\sqrt{6}$ is $2 \cdot 449489$ or $2 \cdot 449490$, to six places

The upper limit $1/2x^2\sqrt{N}$ is hardly obvious, but the reader can derive it without a great deal of trouble.

The two limits just used have a connection with continued fractions which will now be pointed out. The $\sqrt{N}$ has a simple continued fraction expansion of the form $a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \ldots$. We shall denote the nth convergent, as usual, by p_n/q_n, and the number of partial quotients in the period or cycle of the expansion by c. It is then well known that, when c is even, the complete solution of $y^2 - Nx^2 = 1$ is given by $y = p_{tc}$, $x = q_{tc}$, t being a positive integral parameter, and that when c is odd, the general solution is given by $y = p_{2tc}$, $x = q_{2tc}$. Hence the above approximation is essentially a continued fraction method, even if we happen to find our solutions of $y^2 - Nx^2 = 1$ by some other method.

On the other hand, if we were approximating to a square root by the continued fraction method, we could still use the limits obtained above and would find it advantageous to do so. The usual treatments show, that the error committed in taking p_n/q_n as the true value of the continued fraction is less than $1/q_n q_{n+1}$ and greater than $a_{n+2} \div q_n q_{n+2}$. These limits show that $49/20$, as an approximation to $\sqrt{6}$, is in error by more than $1/1980$ and less than $1/1780$. These are poorer limits than the others, and the same situation almost always obtains. We thus state the theorem

Theorem. If c is even, the error in taking p_{tc}/q_{tc} as the value of N, or if c is odd, the error in taking p_{2tc}/q_{2tc} as the value, is greater than $1/2\,pq$ but less than $1/2q^2\sqrt{N}$.*

* Subscripts are omitted here for convenience.

The same limits actually apply to the error in taking p_{tc}/q_{tc}, t odd, as the value when c is odd, but the upper and lower limits are interchanged. I shall not take the space to prove this.

From the standpoint of actual computation, the approximation method of Jnānarāja, given first by Heron about 200 A.D., is superior to the Diophantine method of Nārāyana. We shall now take Heron's method, and modify it very slightly so that our theorem on errors can still be used. Heron and Jnānarāja, to approximate $\sqrt{N}$, find the nearest perfect square a^2 to N, and form the closer approximation $a_1=\frac{1}{2}\left(a+\frac{N}{a}\right)$. A next approximation $a_2=\frac{1}{2}\left(a_1+\frac{N}{a}\right)$ is formed, and so on. It is obviously not necessary to choose a in quite the way stated; it may be any fairly close approximation. A perfectly natural way to get a close first approximation is by continued fractions, since it is well-known that any convergent is the best approximation of all fractions that have their denominators not greater than that of the convergent. Say we agree to take as our first approximation p_c/q_c. Now an interesting theorem given by Chrystal, in Volume II of his Algebra, page 440, shows that

$$\frac{p_{2tc}}{q_{2tc}}=\frac{p^2_{tc}+Nq^2_{tc}}{2p_{tc}\,q_{tc}}$$

But the right-hand side may be written as $\frac{1}{2}\left(\frac{p_{tc}}{q_{tc}}+\frac{Nq_{tc}}{p_{tc}}\right)$. We see then that Heron's method, with $a=p_c/q_c$ gives $a_1=p_{2c}/q_{2c}$, $a_2=p_{4c}/q_{4c}$, and so on. Each p/q in this series with the exception of p_c/q_c when c is odd, is a solution of $y^2-Nx^2=1$; so we have a relation between this method and the Diophantine method of approximation. However, Heron's method is very easy to use and skips over many of the solutions of the Diophantine equation. This makes it better to use than the Diophantine equation method, even when the Euler recurrence formulas are used to give the successive solutions.

Our theorem on limits of error still applies, and gives the most satisfactory way of estimating the error in an approximation by Heron's method, with the first approximation chosen as above. It may be interesting to mention that in recent years Bouton (*Annals*

of Mathematics, ser. 2, Vol. 10, 1908-9, 167-72) and James (*American Mathematical Monthly*, Vol. 31, 1924, 471-5) have studied Heron's method and determined error expressions. Neither gives both an upper and lower limit.

In conclusion, we have considered two methods of approximating to the square root of N, and have seen their relation to each other as well as to the ordinary continued fraction method. We have found that one of the methods is the more advantageous to use, but that it may well be used in connection with error expressions obtained in connection with the other method. This last fact is the main contribution of the paper.

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NOTICE

Krishna Kumari Ganesh Prasad Medal and Prize.

First award to be made in 1936.

The Council of the Calcutta Mathematical Society has chosen the following subject for the thesis to the author of which a gold medal and a cash prize of Rs. 200 will be awarded in January, 1936:—

The lives and the works of the ten Hindu Mathematicians: Varāha-Mihira, Āryabhaṭa, Bhāskara I, Brahmagupta, Lalla, Srīdhara, Mahāvīra, Srīpati, Bhāskara II and Nārāyaṇa.

The rules for the competition were published in the *Bulletin of the Calcutta Mathematical Society*, Vol. 22, Nos. 2 & 3, 1930, and they are reproduced below for ready reference.

(1) A research prize and gold medal shall be instituted to be named Krishna Kumari Ganesh Prasad Prize and Medal after the name of the donor's daughter.

(2) The prize and the medal shall be awarded every fifth year to the author of the best thesis embodying the result of original research or investigation in a topic connected with the history of Hindu Mathematics before 1600 A.D.

(3) The subject of the thesis shall be prescribed by the Council of the Calcutta Mathematical Society at least two years in advance.

(4) The last day of submitting the thesis for the award in a particular year shall be the 21st March preceding that year.

(5) The prize and the medal shall be open to competition to all nationals of the world without any distinction of race, caste or creed.

(6) A board of Honorary Examiners, consisting of (*i*) the President of the Society, (*ii*) an expert in the subject nominated by the donor, or after his death, such an expert nominated by the donor's heirs, and (*iii-v*) three experts in the subject elected by the Council of the Society, shall be appointed as soon as possible after the last day of receiving the theses.

(7) The recommendation of the Board of Examiners shall be placed before the next annual meeting of the Society and the decision of that meeting shall be final.

(8) Every candidate shall be required to submit three copies (type written) of his or her thesis.

(9) If in any year no thesis is received or the theses submitted be pronounced by the Board of Examiners to be not of sufficient merit, a second prize or a prize in a second subject, or a prize of enhanced value, may be awarded in a subsequent year or years as the Council of the Calcutta Mathematical Society may determine.

(10) The thesis of the successful candidate shall be printed by the Society.

EXTRACTS

FROM THE

CONSTITUTION OF THE SOCIETY

10 Every person, desirous of admission into the Society as a Member, must be proposed and recommended by at least two members.

11 Every recommendation of a proposed Member must be delivered to the Secretary and read at one of the ordinary meetings of the Society. * * * * * * * *

14 Every person elected as a Member, shall pay his admission fee within one month, and the first annual contribution within three months of the day of his election; otherwise his election shall be void; but the Council shall, in particular cases, have the power of extending the period within which the first annual contribution shall be paid.

15 * * The admission fee shall be ten rupees in all cases; the annual contribution shall be twelve rupees in the case of Members residing in Calcutta or its Suburbs and Howrah (who shall be called Resident Members), and six rupees in the case of Non-resident Members. * * * * *

16 Members who have paid the admission fee and a sum of one hundred rupees in advance shall become Life Members.

Authors of Papers intended for communication to the Society are requested to draw up Abstracts of their Papers, to be read at the meetings of the Society and to be circulated if possible, the Abstracts to indicate the nature of the methods employed and the character of the results obtained.

A Paper accepted for publication in the Bulletin of the Society should not be re-published by the author in a different Journal without the sanction of the Council and without making the necessary acknowledgment.

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June, 1932

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